Faster Wavelet Tree Queries

Data Compression Conference (DCC 2024)

Matteo Ceregini, *Florian Kurpicz*, and Rossano Venturini

Operations on Sequences

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a	c	c	e	s	s	a	n	d	s	e	l	e	c	t

Applications

- compression
- computational geometry
- pattern matching
- ...

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$\mathrm{rank}_s(10) = 3$

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$\mathrm{rank}_s(10) = 3$

$\mathrm{select}_s(3) = 9$

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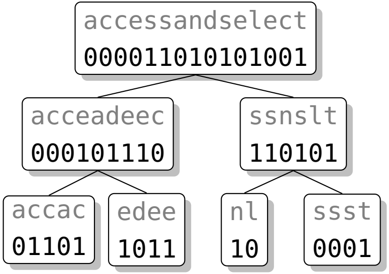
- compression
- computational geometry
- pattern matching
- ...

Wavelet Trees [GGV03]



- de-facto standard for access, rank, and queries
- $O(\log \sigma)$ query time
- require $\lceil H_0(T) \rceil n(1 + o(1))$ bits of space

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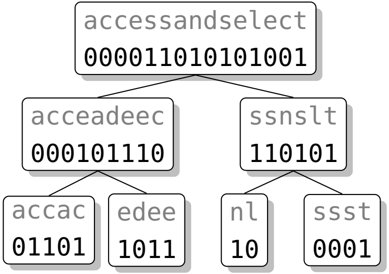

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Previous Work

- lots of work on construction [Bab+15; CNP15; DFK20; Din+21; Din+23; EK19; Fue+17; Kan18; LSB17; MNV16; Shu20]
- little work on queries [CNP15; Fer+07]

Faster Queries @ DCC'24

- "Faster Wavelet Tree Queries " (this paper)
- "Another Virtue of Wavelet Forests" (poster)

Wavelet ~~Trees~~ Matrices [CNP15]

```
accessandselect
000011010101001
```

$Z[0] = 9$

```
acceadeecssnslt
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- “everything” known for trees applies to matrices

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Construction

- bit vector on each level
- on *k*-th level symbols represented by *k*-th MSB
- stably sort sequence using written bit as key
- continue with next level
- store number of zeros on each level in Z

Rank Queries

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rank$_\alpha$*(i)*

$r_0 = i,\ b_0 = 0$

for $k = 0, \ldots, \ell$ **do**

$\alpha_k = (\alpha \ >> \ (\ell - 1 - k))\ \&\ 1$

$\texttt{offset} = \alpha_k * Z[k]$

$b_{k+1} = bv[k].rank_{\alpha_k}(b_k) + \texttt{offset}$

$r_{k+1} = bv[k].rank_{\alpha_k}(r_k) + \texttt{offset}$

return $r_{\ell+1} - b_{\ell+1}$

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Cache Misses on Each Level

- binary rank and select queries are expensive
- rank/select data structures not in cache

4-Ary Wavelet Matrices

- use quad vectors instead of bit vectors
- space overhead 3.51 % $\rightsquigarrow$ 6.25 %
- rank possible with 2.41 % space overhead

- $\lceil \log \sigma / 2 \rceil$ levels (uncompressed)
- halve cache misses for rank/select data structures

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$C_i[\alpha]$ number elements $< \alpha$ in level i

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- more for rank queries in the tree/matrix
- path through tree/matrix known at query time

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Rank with Additive Approximation (RAA)

Karlsruhe Institute of Technology

Definition. Let $Q[1, n]$ be a quad vector and $\epsilon \in \mathbb{N}$.
The RAA for a position *i* and a symbol $\alpha \in [0, 3]$ is

$$rank_\alpha(i) \in [rank^{\approx}_\alpha(i), rank^{\approx}_\alpha(i) + \epsilon).$$

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Proof (Sketch). Split quad vector $Q[1..n]$ into blocks of size $\epsilon/2$.

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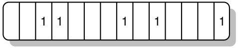
B_α 

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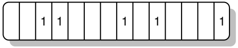

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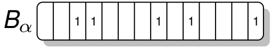

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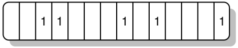

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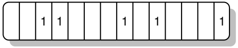
B_α 

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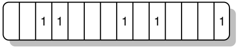

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Lemma. The RAA for quad vectors can be solved in constant time using $\Theta(n/\epsilon)$ bits of space.

Lemma. A RAA data structure for quad vectors requires $\Omega(n/\epsilon)$ bits of space.

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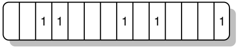
B_α 

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Predicting Cache Lines in the Wavelet Tree

Problem

- let r_k be the rank on the k-th level
- RAA does not guarantee that $r_k \in [r_k^{\approx}, r_k^{\approx} + \epsilon)$
- we can only compute $rank_{\alpha_k}^{\approx}(r_{k-1}^{\approx})$
- error could be up to $(k-1)\epsilon$

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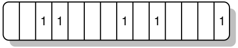
B_α 

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- store in $D_{k,\alpha}$
- use rank/select on bit vector to access offset

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- let d_{k-1} be successor of r_{k-1} in D_{k,α_k}
- $\Delta = \min(d_{k-1} - r_{k-1}^{\approx}, \epsilon - 1)$
- $r_k^{\approx} = rank_{\alpha_k}(d_{k-1}) - \Delta$

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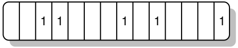
B_α 

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Lemma. At any level k, we have $r_k \in [r_k^{\approx}, r_k^{\approx} + \epsilon)$.

- requires $\Theta((n/\epsilon \log \sigma) \log \epsilon)$ bits of space
- problematic if predictor does not fit into cache
- use hierarchy of predictors

Practical Implementations and Experiments

- Δ and $D_{k,\alpha}$ not necessary
- error always small enough for σ up to 256
- prefetch more cache lines

- use two levels of predictors
- first level with $\epsilon = 2048$
- second level with $\epsilon = 256$
- help to prefetch blocks and super blocks

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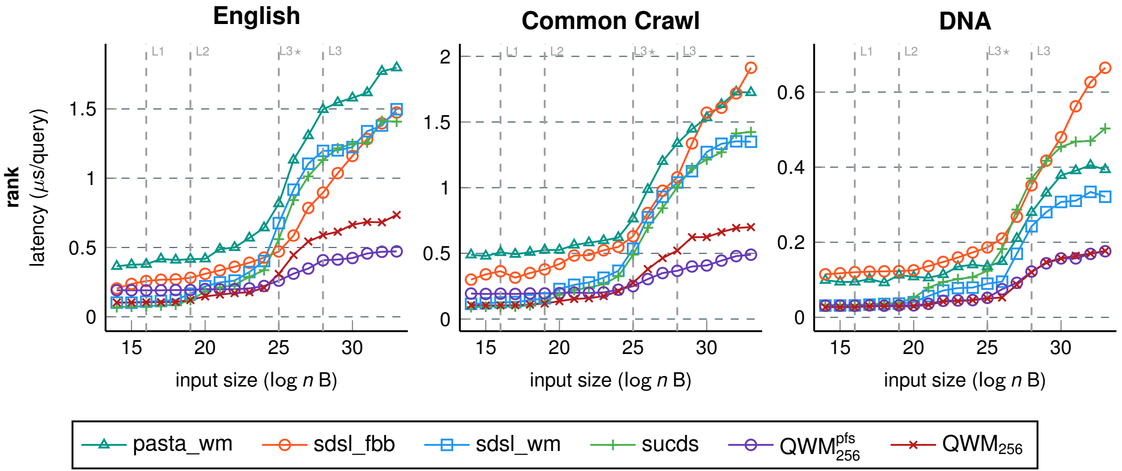
Experimental Setup

- AMD EPYC 7713
  - 64 KB L1I and 64 KB L1D per core
  - 512 KB L2I+D per core
  - 256 MB L3I+D (32 MB per 8 cores CCX)
- 2 TB DDR4 RAM
- Ubuntu 20.04.3 LTS kernel version 5.4.0-155
- C++: GCC 11.1.0 (`-O3 -march=native`)
- Rust: `cargo build -release`

Experimental Evaluation (Latency)



Conclusion and Future Work

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This Paper

- up to 3 times faster wavelet tree queries
- predictive model for rank queries

What's Next

- compressed wavelet trees
- use predictive model for other data structures

Check It Out

- `https://github.com/rossanoventurini/qwt`

European Research Council
Established by the European Commission

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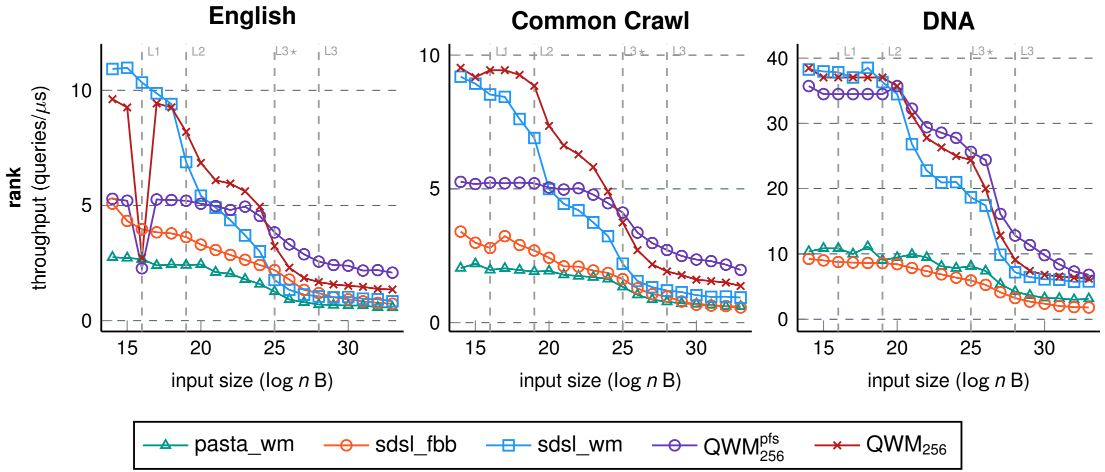
Experimental Evaluation (Throughput)



Bibliography I

[Bab+15] Maxim A. Babenko, Pawel Gawrychowski, Tomasz Kociumaka, and Tatiana Starikovskaya. "Wavelet Trees Meet Suffix Trees". In: *SODA*. SIAM, 2015, pages 572–591. DOI: 10.1137/1.9781611973730.39.

[CNP15] Francisco Claude, Gonzalo Navarro, and Alberto Ordóñez Pereira. "The Wavelet Matrix: An Efficient Wavelet Tree for Large Alphabets". In: *Inf. Syst.* 47 (2015), pages 15–32. DOI: 10.1016/j.is.2014.06.002.

[DFK20] Patrick Dinklage, Johannes Fischer, and Florian Kurpicz. "Constructing the Wavelet Tree and Wavelet Matrix in Distributed Memory". In: *ALENEX*. SIAM, 2020, pages 214–228. DOI: 10.1137/1.9781611976007.17.

[Din+21] Patrick Dinklage, Jonas Ellert, Johannes Fischer, Florian Kurpicz, and Marvin Löbel. "Practical Wavelet Tree Construction". In: *ACM J. Exp. Algorithmics* 26 (2021), 1.8:1–1.8:67. DOI: 10.1145/3457197.

Bibliography II

[Din+23] Patrick Dinklage, Johannes Fischer, Florian Kurpicz, and Jan-Philipp Tarnowski. “Bit-Parallel (Compressed) Wavelet Tree Construction”. In: *DCC*. IEEE, 2023, pages 81–90. DOI: 10.1109/DCC55655.2023.00016.

[EK19] Jonas Ellert and Florian Kurpicz. “Parallel External Memory Wavelet Tree and Wavelet Matrix Construction”. In: *SPIRE*. Volume 11811. Lecture Notes in Computer Science. Springer, 2019, pages 392–406. DOI: 10.1007/978-3-030-32686-9_28.

[Fer+07] Paolo Ferragina, Giovanni Manzini, Veli Mäkinen, and Gonzalo Navarro. “Compressed representations of sequences and full-text indexes”. In: *ACM Trans. Algorithms* 3.2 (2007), page 20. DOI: 10.1145/1240233.1240243.

[Fue+17] José Fuentes-Sepúlveda, Erick Elejalde, Leo Ferres, and Diego Seco. “Parallel construction of wavelet trees on multicore architectures”. In: *Knowl. Inf. Syst.* 51.3 (2017), pages 1043–1066.

[GGV03] Roberto Grossi, Ankur Gupta, and Jeffrey Scott Vitter. “High-Order Entropy-Compressed Text Indexes”. In: *SODA*. ACM/SIAM, 2003, pages 841–850.

Bibliography III

[Kan18] Yusaku Kaneta. “Fast Wavelet Tree Construction in Practice”. In: *SPIRE*. Volume 11147. Lecture Notes in Computer Science. Springer, 2018, pages 218–232. DOI: 10.1007/978-3-030-00479-8_18.

[LSB17] Julian Labeit, Julian Shun, and Guy E. Blelloch. “Parallel lightweight wavelet tree, suffix array and FM-index construction”. In: *J. Discrete Algorithms* 43 (2017), pages 2–17. DOI: 10.1016/j.jda.2017.04.001.

[MNV16] J. Ian Munro, Yakov Nekrich, and Jeffrey Scott Vitter. “Fast construction of wavelet trees”. In: *Theor. Comput. Sci.* 638 (2016), pages 91–97. DOI: 10.1016/j.tcs.2015.11.011.

[Shu20] Julian Shun. “Improved parallel construction of wavelet trees and rank/select structures”. In: *Inf. Comput.* 273 (2020), page 104516. DOI: 10.1016/j.ic.2020.104516.