On Maximum Common Subgraph Problems in Series-Parallel Graphs

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Output: Maximum number of vertices in a connected graph that is isomorphic to an induced subgraph of G and H.

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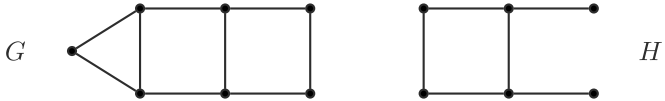

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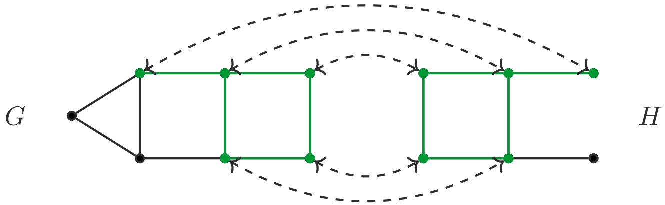

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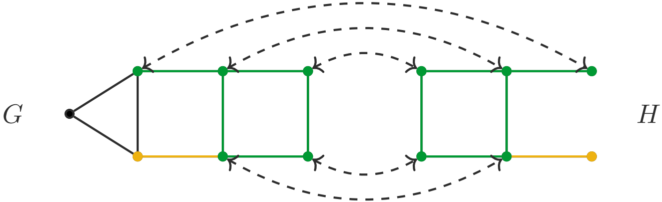

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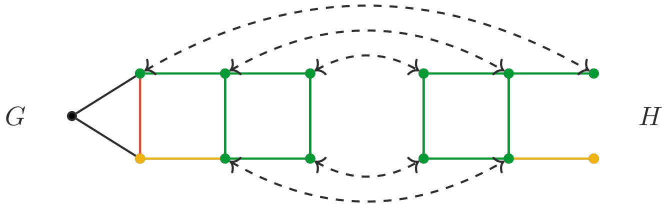

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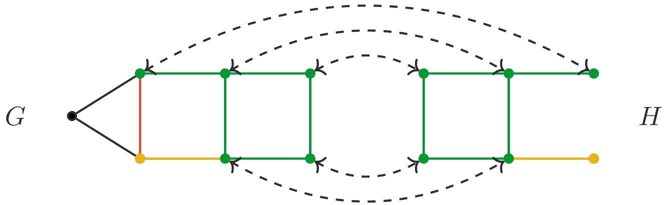

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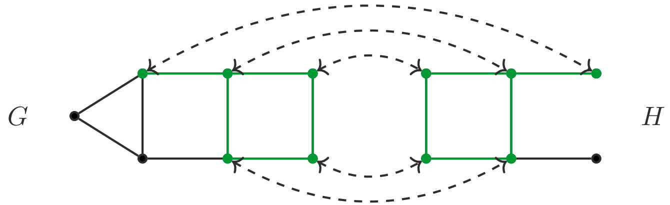

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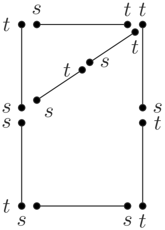

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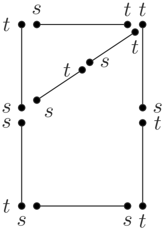

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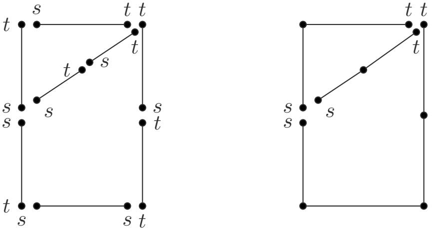

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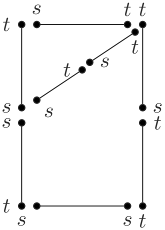

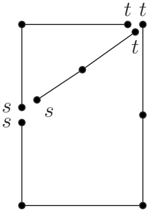

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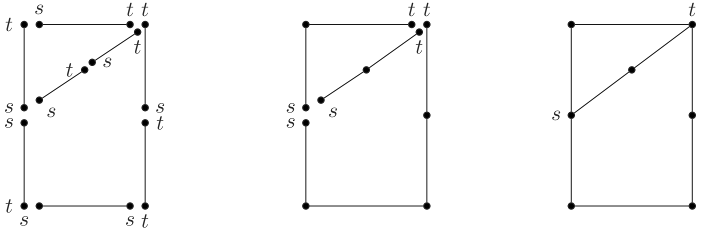

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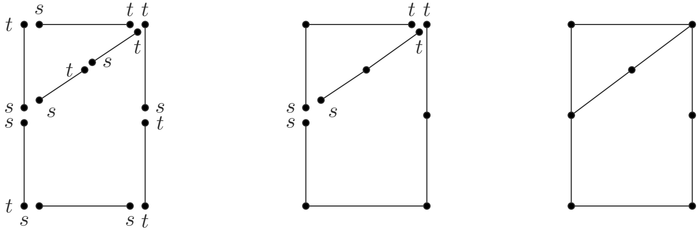

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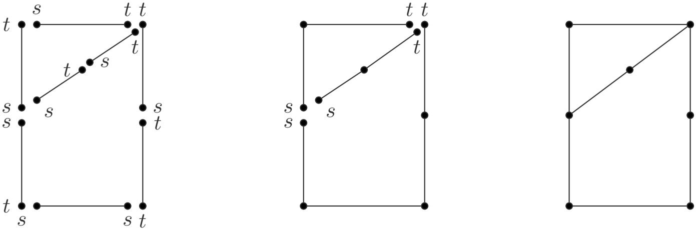

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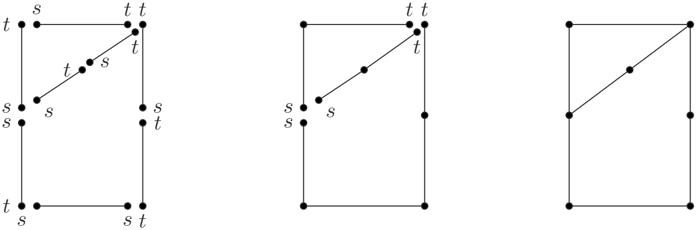

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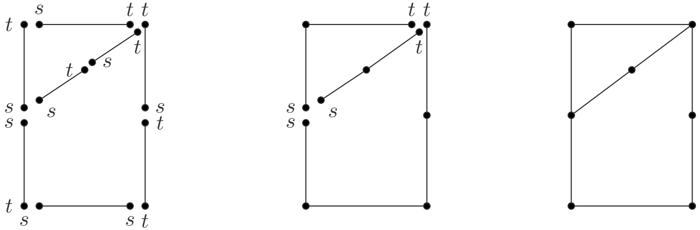

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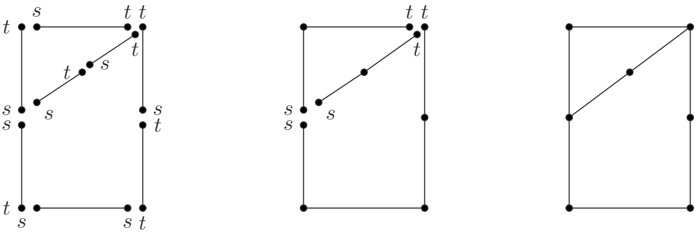
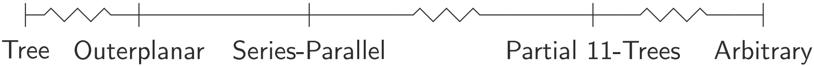


Lemma [Brandstadt et al., 1999]

G is series-parallel $\iff$ G is a partial 2-tree.

The Complexity of the Problem



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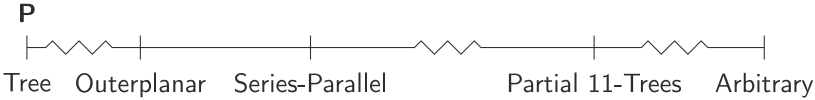

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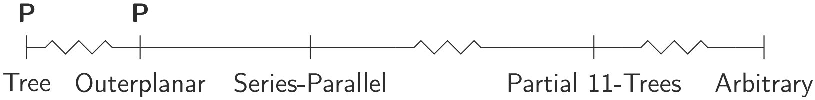

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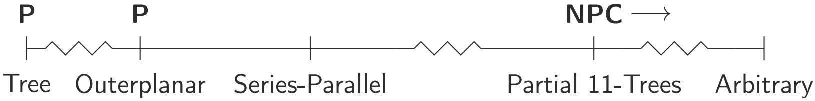

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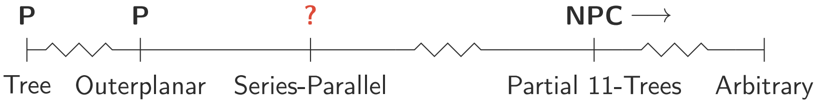

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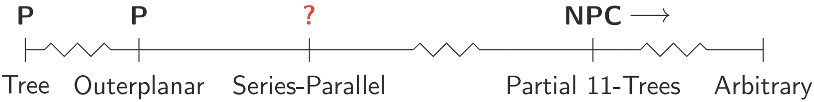

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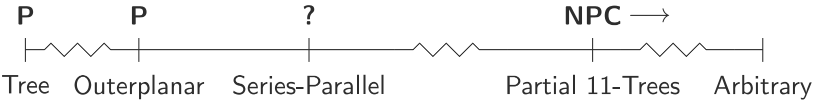

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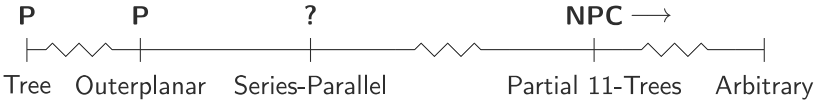

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Theorem [Gupta & Nishimura 1994]

SI in partial k-trees with degree $\leq k+2$ for all but k vertices is in **NPC**.

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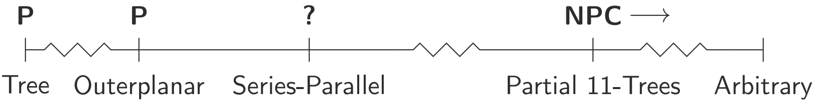

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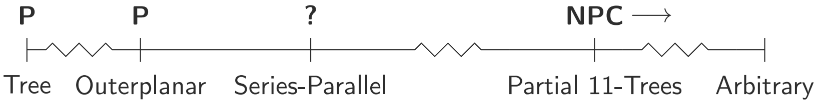

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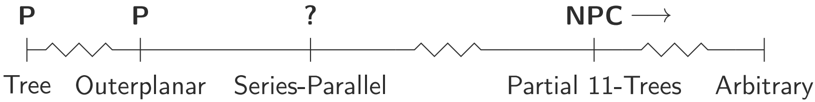

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$\text{MCIS}^{\leq 3,1}$: **NP**-hard → now

Reduction from the . . .

Definition (Numerical Matching with Target Sums (NMwTS))

Input: Two multisets of integers X and Y with $|X| = |Y| = n$ and a vector $\vec{b} = \langle b_1, \ldots, b_n \rangle$ with $b_i \in \mathbb{N}_0$ for all $i = 1, \ldots, n$.

Question: Can $X \cup Y$ be partitioned into disjoint sets $A_1, \ldots, A_n$ each containing one element from X and Y, such that $\sum_{a \in A_i} v(a) = b_i$ for all $i = 1, \ldots, n$?

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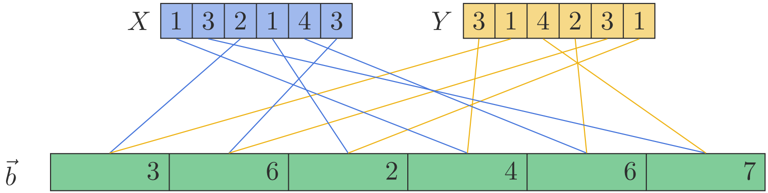

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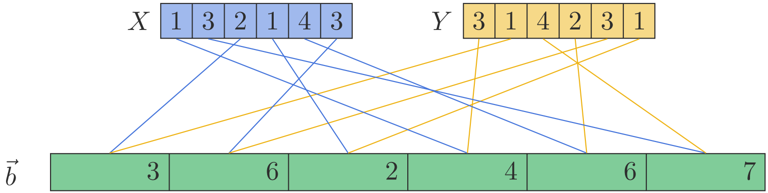

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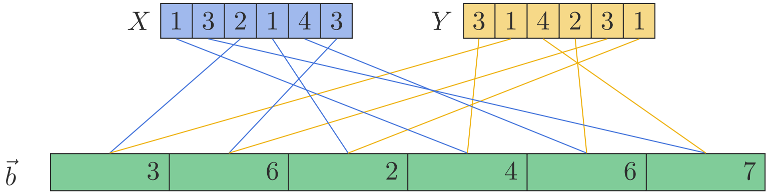

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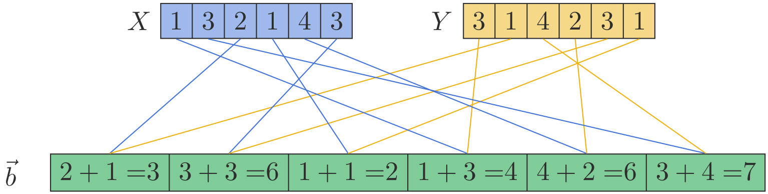

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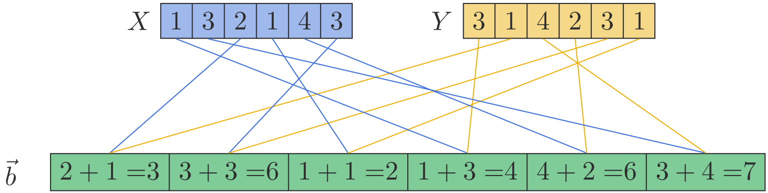

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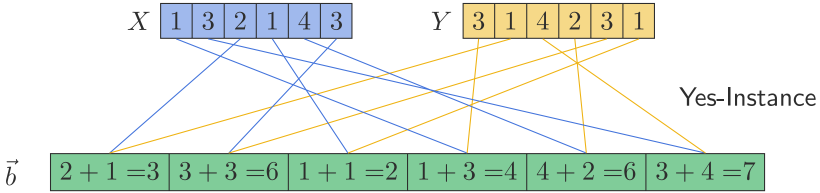

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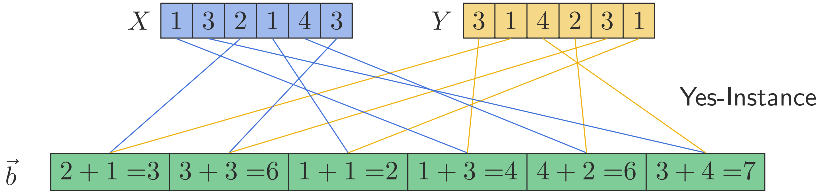

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- **NPC** in the strong sense $\iff$ **NPC** with unary encoded input.

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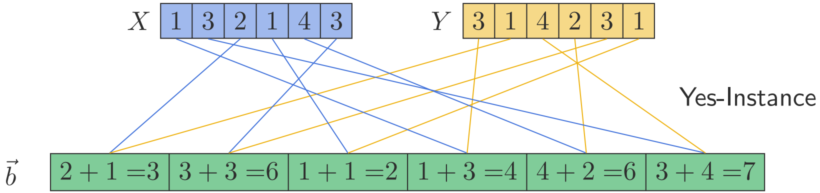

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- **NPC** in the strong sense $\iff$ **NPC** with unary encoded input.
- NMwTS is **NPC** in the strong sense.

Construction of the Reduction

Outline for each NMwTS-Instance $(X, Y, s, \vec{b})$

- Represent X, Y and $\vec{b}$ as graphs G and H.
- Size of MCIS indicates type of instance.

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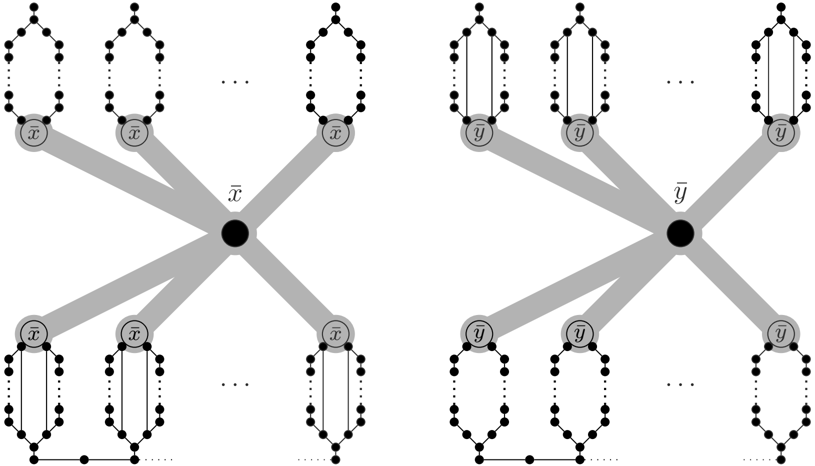
Parts of the Graphs

1. Base-Gadgets $\rightarrow$ Structure of the MCIS.
2. Encoding of the sizes of X, Y and $\vec{b}$ $\rightarrow$ Size of the MCIS.

The Base-Gadgets of G and H



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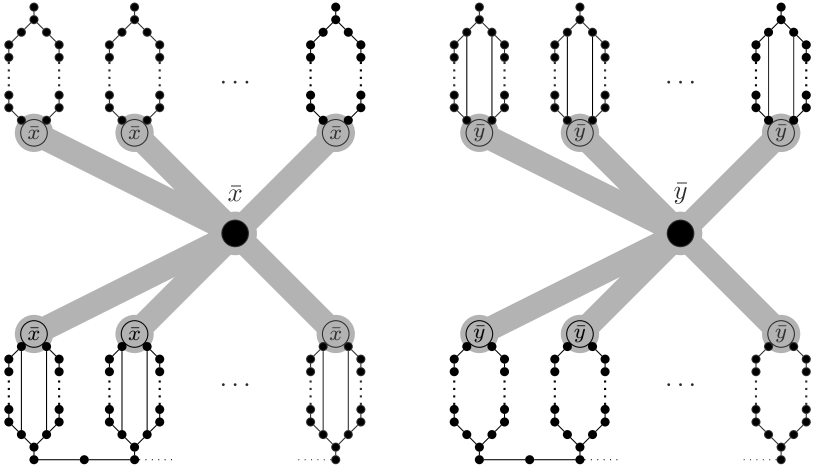

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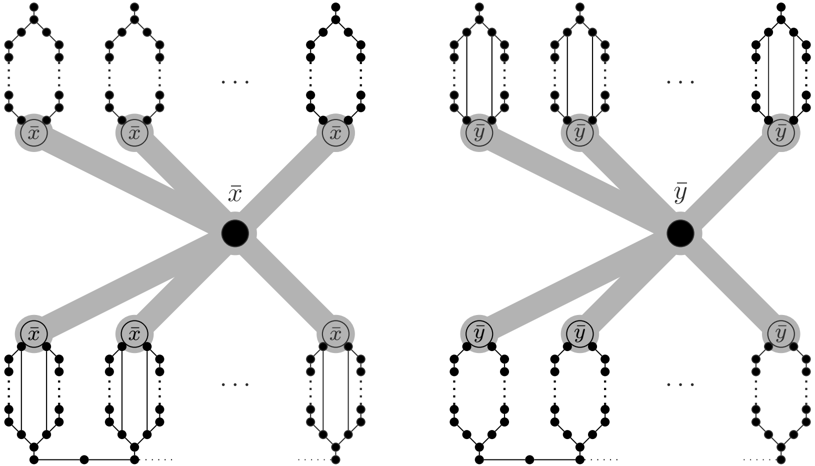

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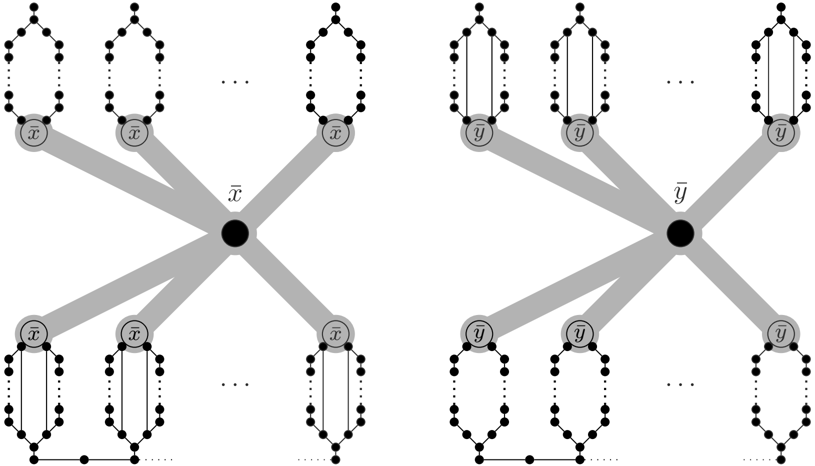

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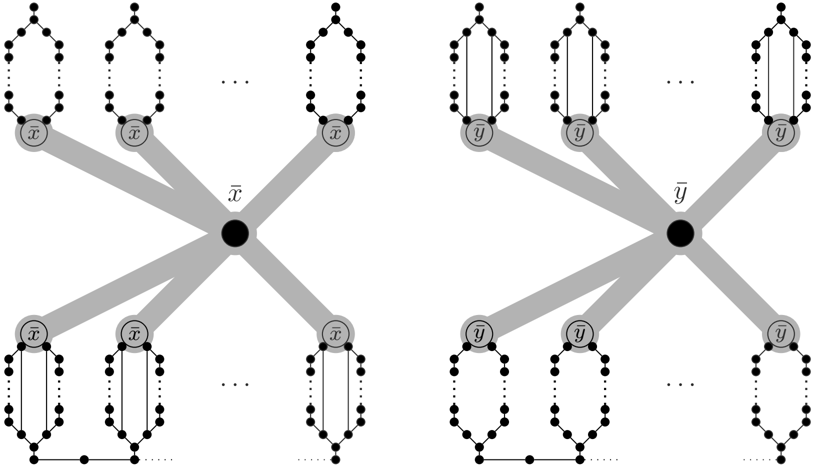

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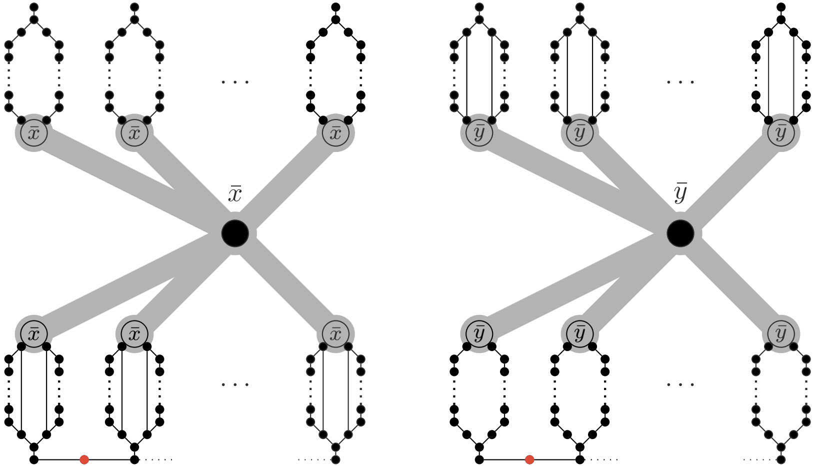

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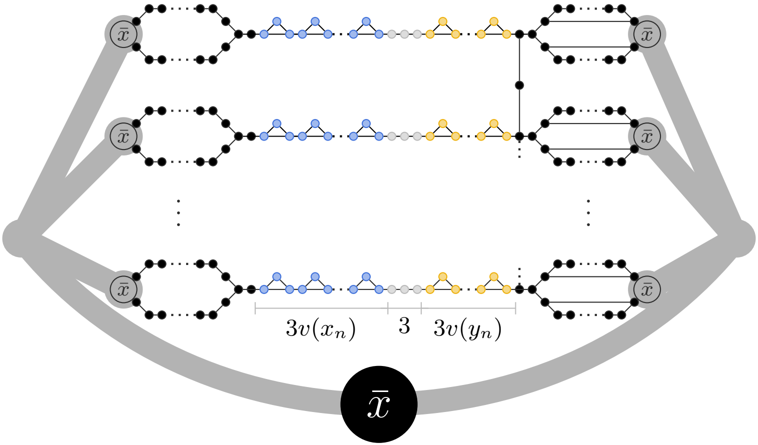
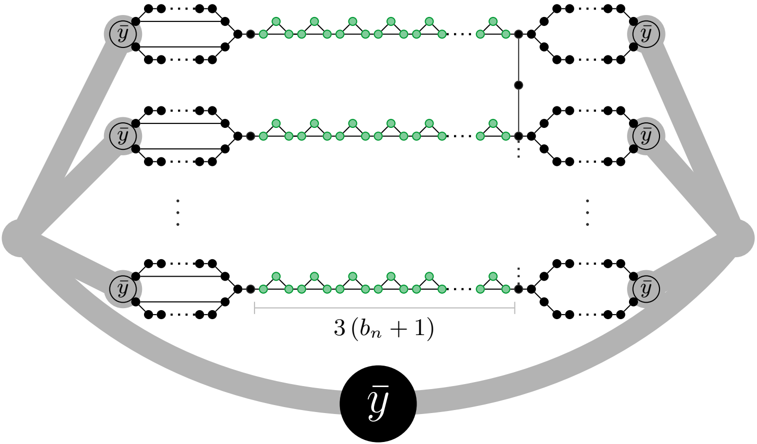


- Values $v(x_i)$ and $v(y_i)$
- Separating paths

Encode the Sizes of $\vec{b}$



- Entries of $\vec{b}$
- $+1$ to compensate the separating paths

The Reduction (Sketch)

Theorem

MCIS in SPGs with degree ≤ 3 for all but 1 vertex is **NP**-hard.

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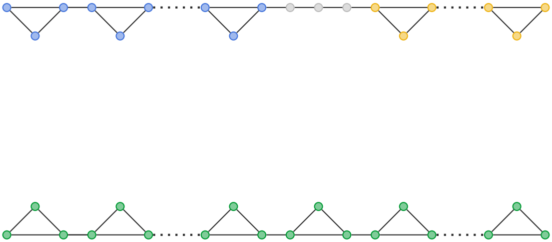

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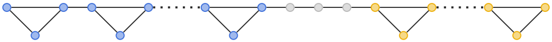

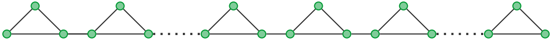

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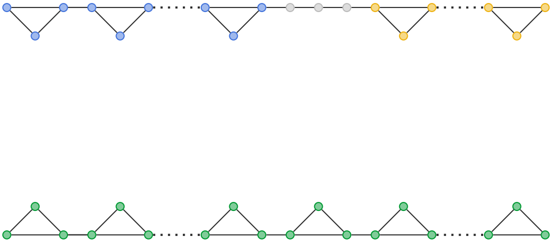

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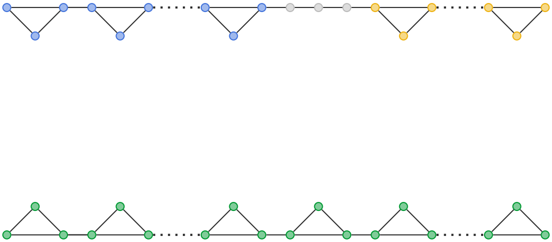

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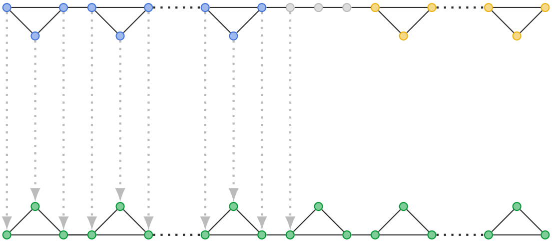

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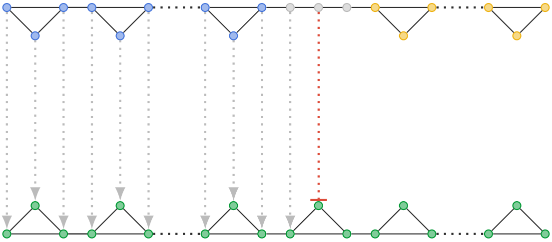

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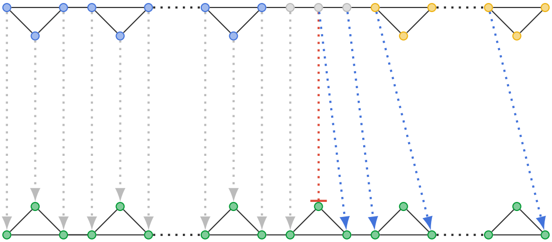

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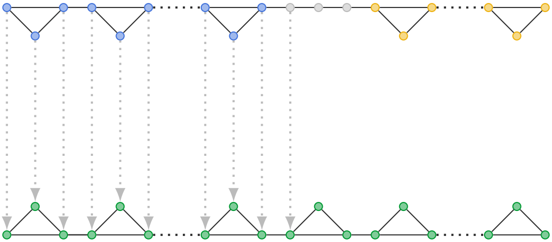

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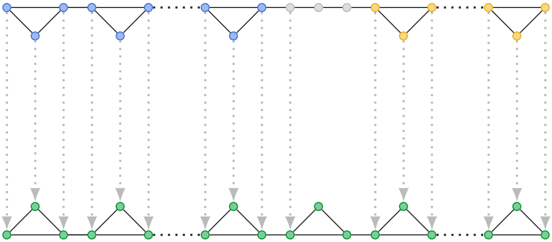

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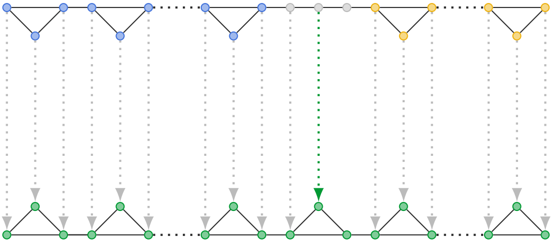

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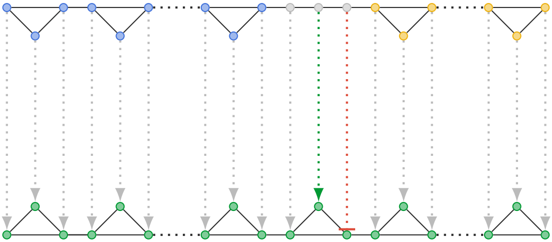

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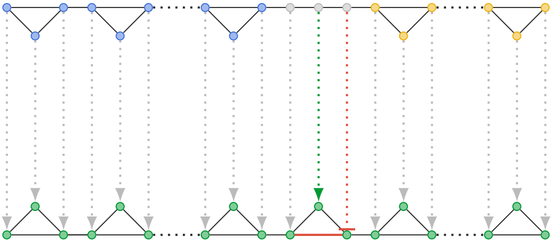

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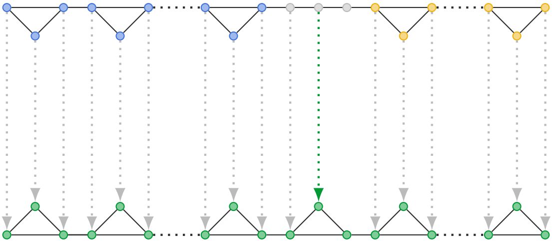

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A Polynomial Time Algorithm

Definition (k-Maximum Common Subgraph (k-MCS))

Input: k-connected partial k-trees G, H.

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Known Variants of k-MCS in P

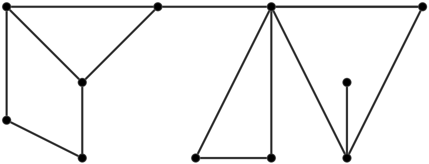
1-MCS $\Leftrightarrow$ Maximum Common Subtree Problem ✓ [Matula, 1978]

2-MCIS in SPGs ✓ [Kriege & Mutzel, 2014]

Graph Decomposition

Definition (BC-Tree)

- Tree consisting of a node for each cut vertex and block.
- Distinguish between bridges and non-bridge blocks.
- Edges between nodes if cut vertex is contained in block.



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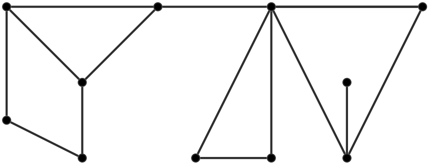

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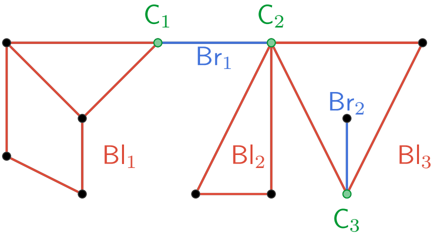

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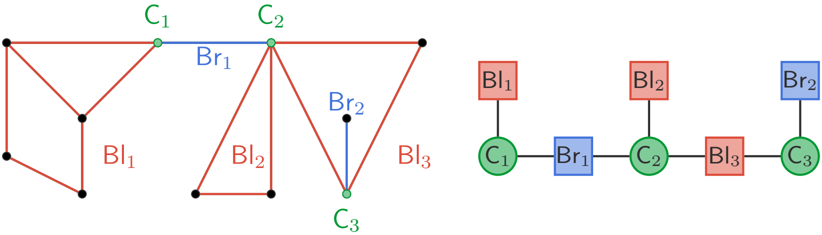

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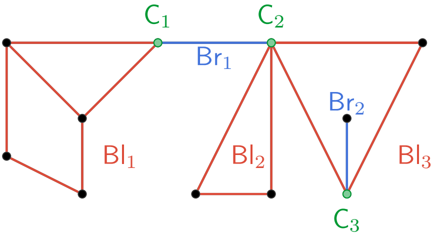

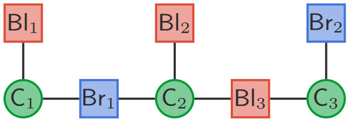

- BC-tree is bipartite with respect to cut vertices and blocks.
- Non-bride blocks represent 2-connected SPGs.

Fewer Restrictions

Definition (Block-and-Bridge Preserving MCS (BBP-MCS))

- Variant of the general common subgraph problem [Schietgat et al., 2007]
- Non-Bridge blocks are only mapped onto non-bridge blocks.
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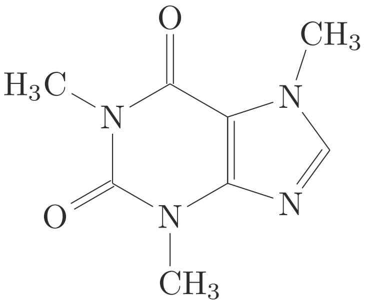

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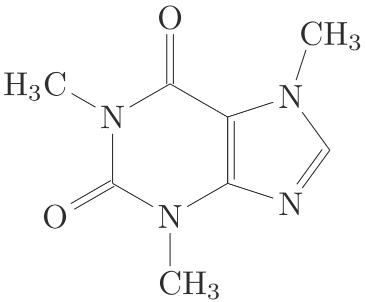

- Only compare blocks of the same type.
- There is an algorithm for 2-MCIS.

Idea of the Algorithm

Edmonds-Matula Algorithm for 1-MCS (Sketch)

1. Decompose trees into rooted subtrees.
2. Compute MCS of subtrees (*Maximum Weighted Bipartite Matching*).
3. Combine partial solutions.

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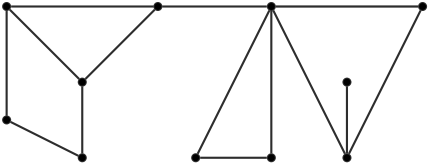
Algorithm for BBP-MCIS (Sketch)

1. Decompose BC-trees into rooted BC-subtrees.
2. Compute MCS of subgraphs represented by subtrees:
   - *Maximum Weighted Bipartite Matching*
   - Extended 2-MCIS algorithm → work with cut vertices
3. Combine partial solutions.

Rooting a BC-Tree

- The root is a vertex in the input graph.
- A subgraph can now be induced by a BC-subtree.



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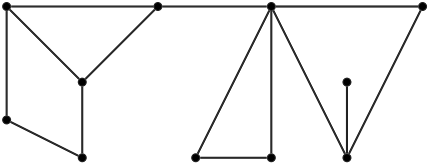

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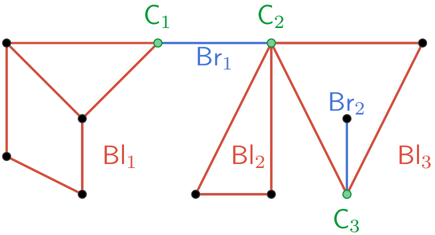

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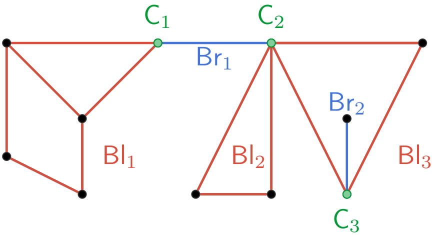

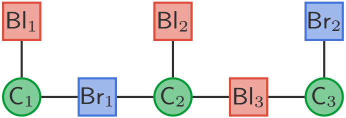

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Proof (Sketch)

- (Extended) 2-MCIS can be solved in $\mathcal{O}\left(n^6\right)$.
- Comparison of all blocks at cut vertices in $\mathcal{O}\left(n^5\right)$ $\rightarrow$ at most $\mathcal{O}\left(n^2\right)$ pairs of cut vertices and MWBM is solvable in $\mathcal{O}\left(n^3\right)$.
- Total time $\rightarrow \mathcal{O}\left(n^6\right)$.

Analysis

Polynomial Time Algorithm

Theorem

BBP-MCIS in SPGs can be solved in $\mathcal{O}\left(n^6\right)$, where $n = |V(G)| + |V(H)|$.

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Theorem

BBP-MCIS in outerplanar graphs can be solved in $\mathcal{O}\left(n^5\right)$.

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Theorem

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Proof (Sketch)

- (Extended) 2-MCIS in outerplanar graphs can be solved in $\mathcal{O}(n^3)$.
- Comparison of all blocks stays expensive.

Conclusion & Future Work

- **NP**-hardness of MCIS in SPGs with one vertex of unbounded degree.
- BBP-MCIS in SPGs can be solved in polynomial time.

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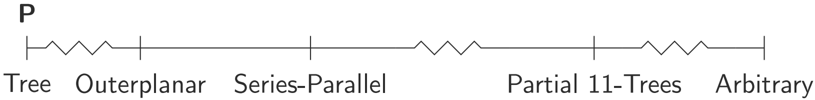

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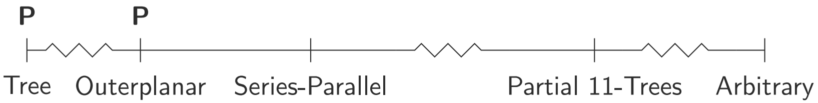

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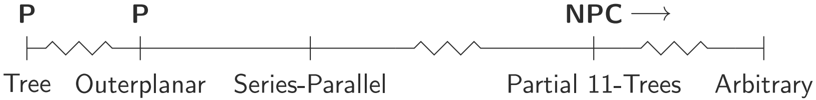

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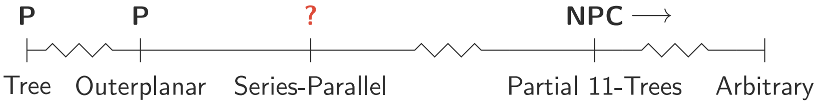

Open Questions

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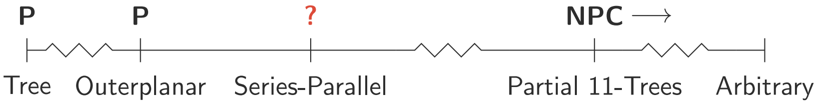

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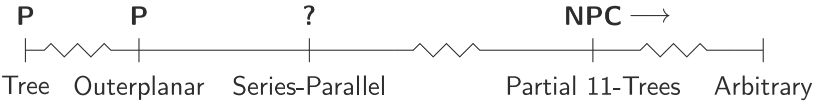

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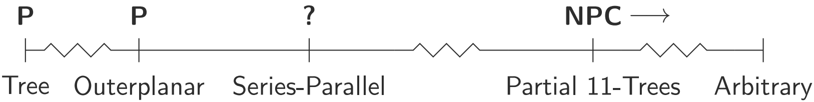

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Thank You!