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Advanced Data Structures

Lecture 07: Compressed Suffix Array

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PINGO

https://pingo.scc.kit.edu/202619

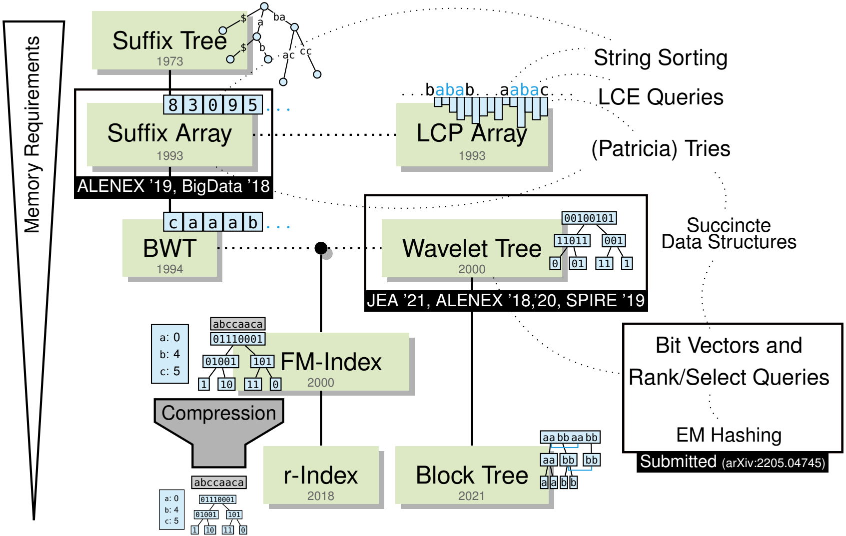
Recap: Suffix Array

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

- space: $O(n \log n) bits$
- space text: $n\lceil \log \sigma \rceil$ bits

- better: index requiring same space as text
- even better: index requiring same space as compressed text

(Compressed) Text Indices #Ad



Ψ Function

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Definition: Ψ Function

Given a suffix array *SA* of length *n*,

$$\Psi(i) = SA^{-1}[SA[i] + 1]$$

- $SA[\Psi(i)] = SA[i] + 1$
- where in *SA* is the suffix $T[SA[i+1]..n)$
- "successor" function

- can be used to obtain suffix array
- can be compressed ℹ currently $O(n \log n)$ bits

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

Replacing *SA* with Ψ

- which number does in this example not occur?
  Answer: 3
- how to obtain $SA[i]$ using Ψ **PINGO**

- follow positions until last suffix is found
- last suffix is at position 1
- $n - \#steps$ is *SA* value
- requires $O(n)$ time

- pattern matching: $O(mn \log n)$ time
- pattern matching with *LCP* and *RMQ*: $O(mn + \log n)$ time

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

Speeding Up Lookups in Ψ (1/2)

- space SA: $O(n \log n)$ bits
- space text: $O(n \log \sigma)$ bits
- space compressed suffix array should not more than text

- sample every $\log n$-th SA entry
- $O(n/ \log n)$ samples of size $O(\log n)$ bits
- total space: $O(n)$ bits

- every $\log n$-th entry in Ψ
- every $\log n$-th step in Ψ
- what is better? **PINGO**

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	\$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
	\$	a\$	ababcabcabba\$	abba\$	abcabba\$	abcabcabba\$	ba\$	babcabcabba\$	bba\$	bcabba\$	bcabcabba\$	cabba\$	cabcabba\$

Speeding Up Lookups in Ψ (2/2)

- every $\log n$-th entry in Ψ
- every $\log n$-th step in Ψ
- what is better? **PINGO**

- every $\log n$-th step in Ψ is better
- sampled positions may not be reached in better asymptotic time

- how much time does recovering *SA* position from Ψ require with sampling? **PINGO**
- answer: $O(\log n)$

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

Structure of Ψ

- does Ψ have some structure? **PINGO**

Lemma: Structure of Ψ

$T[SA[i]] = T[SA[i+1]] \Rightarrow \Psi(i) < \Psi(i+1)$

Proof (Sketch)

- $T[SA[i]] \leq T[SA[i+1]]$
- if $T[SA[i]] = T[SA[i+1]]$ then $T[SA[i]+1..n) \leq T[SA[i+1]+1..n)$
- $T[SA[i]+1] = T[\Psi(i)]$
- if suffixes share same character, lexicographical order of suffixes without first character holds

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5

- note that not all increasing intervals belong to the same character
- example on the board

Compressing Ordered Sequences

Δ-Encoding

- store difference between entries
- scanning whole sequence up to value when decoding

Elias-Fano (Lecture 05)

- upper and lower halves
- upper half represented in bit vector ℹ $p_i + i$
- lower half plain bit compressed

- using Elias-Fano is bad for large alphabets
- example on the board

0	1	2	3	4	5	6	7	8	9
0	1	2	4	7	10	20	21	22	32

- 0: 000000
- 1: 000001
- 2: 000010
- 4: 000100
- 7: 000111
- 10: 001010
- 20: 010100
- 21: 010101
- 22: 010110
- 30: 100000

upper: 1110110100011100100

lower: 00 01 10 00 11 10 00 01 10 00

Recap: Elias-Fano Coding Space

Lemma: Elias-Fano Coding

Given an array containing n distinct integers from a universe $\mathcal{U} = [0, n)$, the array can be represented using

$$n(2 + \log\lceil\frac{u}{n}\rceil) \text{ bits}$$

while allowing $O(1)$ access time and $O(\log \frac{u}{n})$ predecessor/successor time

Compressing Sparse Ordered Sequences

- Elias-Fano coding for each increasing interval
- σ many
- only every $1/\sigma$-th entry is set (sparse)

- if there are n entries of universe with size u
- make entries sparse using $q = u/n$
- for each value x store pair $(x/q, x\%q)$

- $u = 512, n = 8, q = 64$
- $(0, 3, 17, 89, 128, 132, 500, 511)$
- $\{0, 0\}, \{0, 3\}, \{0, 7\}, \{1, 25\}, \{2, 0\}, \{2, 4\}, \{7, 52\}, \{7, 63\}$

- store quotient (x/q) using Elias-Fano
- store remainder $(x\%q)$ plain using $\lceil \log q \rceil$ bits

Lemma: Ψ with Elias-Fano

Using Elias-Fano with quotienting, Ψ can be stored using $O(n\sigma)$ bits

- more precise: two additional bits per character

Simple Compressed Suffix Array

- compute Ψ and store samples of SA
- compress Ψ Elias-Fano with quotienting
- binary search on SA ℹ by decoding Ψ

- space: $O(n \log \sigma)$ space
- query time: $O(m \log^2 n)$

	1	2	3	4	5	6	7	8	9	10	11	12	13
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SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

Improving Compressed Suffix Arrays [GV05] (1/5)

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- improve *SA* lookup to log log *n* time
- divide-and-conquer approach
- storing Ψ only for half of the entries
- recurs for the other half

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
NEW	13	1	9	3	11	7	5	1	10	6	7	13	4

- for which values do we store Ψ? **PINGO**

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

Improving Compressed Suffix Arrays (2/5)

- store bit vector marking odd *SA* values
- store only odd *SA* values
- store Ψ for even *SA* values

- store Ψ as before
- Elias-Fano with quotienting
- without sampling

- right half (*SA*) still big
- how to recurs?

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
NEW	13	1	9	3	11	7	5	1	10	6	7	13	4
BV	1	0	1	1	0	1	1	0	0	1	0	)	1

Improving Compressed Suffix Arrays (3/5)

- *SA* half consists only of odd values
- for value x store $(x-1)/2$
- reversible since all values are odd

- $13, 1, 9, 3, 11, 7, 5$
- $6, 0, 4, 1, 5, 3, 2$

- what do we have here? **PINGO**
- permutation ℹ basically a suffix array without text

- recurs on the permutation without explicitly storing it

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
NEW	13	1	9	3	11	7	5	1	10	6	7	13	4
BV	1	0	1	1	0	1	1	0	0	1	0	0	1

Improving Compressed Suffix Arrays (4/5)

- recurs $\log \log n$ times
- guarantees $O(\log \log n)$ time to obtain *SA* value
- allows to store final *SA* within space bounds

Lemma: Space Final *SA*

Using the divide-and-conquer approach, the final *SA* requires $O(n)$ bits of space

Proof (Sketch)

- after $\log \log n$ recursions *SA* has size $n/2^{\log \log n}$
- each entry requires $\log n$ bits
- total space: $O(n)$ bits

	1	2	3	4	5	6	7	8	9	10	11	12	13
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Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
NEW	13	1	9	3	11	7	5	1	10	6	7	13	4
BV	1	0	1	1	0	1	1	0	0	1	0	0	1

Improving Compressed Suffix Arrays (5/5)

Lemma: Decoding Time Improved CSA

An *SA* value can be decoded in $O(\log\log n)$ time using the improved CSA

Proof (Sketch)

- on each level, odd *SA* values can be decoded using the recursive *SA*
- there are at most $\log\log n$ levels
- on each level, even *SA* values can be decoded in one step, as the next *SA* value is odd

- requires rank and select data structures

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Ψ	-	1	8	9	10	11	2	6	7	12	13	4	5
NEW	13	1	9	3	11	7	5	1	10	6	7	13	4
BV	1	0	1	1	0	1	1	0	0	1	0	0	1

Conclusion and Outlook

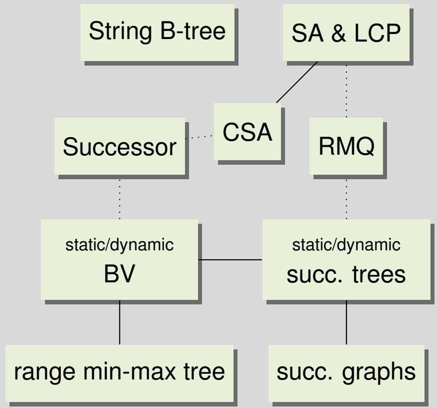
This Lecture

- compressed suffix array
- note that CSA can be compressed further
- Elias-Fano for sparse sequences

Next Lecture

- temporal data structures

Advanced Data Structures



Bibliography I

[GV05] Roberto Grossi and Jeffrey Scott Vitter. “Compressed Suffix Arrays and Suffix Trees with Applications to Text Indexing and String Matching”. In: *SIAM J. Comput.* 35.2 (2005), pages 378–407. DOI: 10.1137/S0097539702402354.