Karlsruhe Institute of Technology

# Advanced Data Structures

## Lecture 08: Compressed Suffix Array

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# PINGO

https://pingo.scc.kit.edu/915810

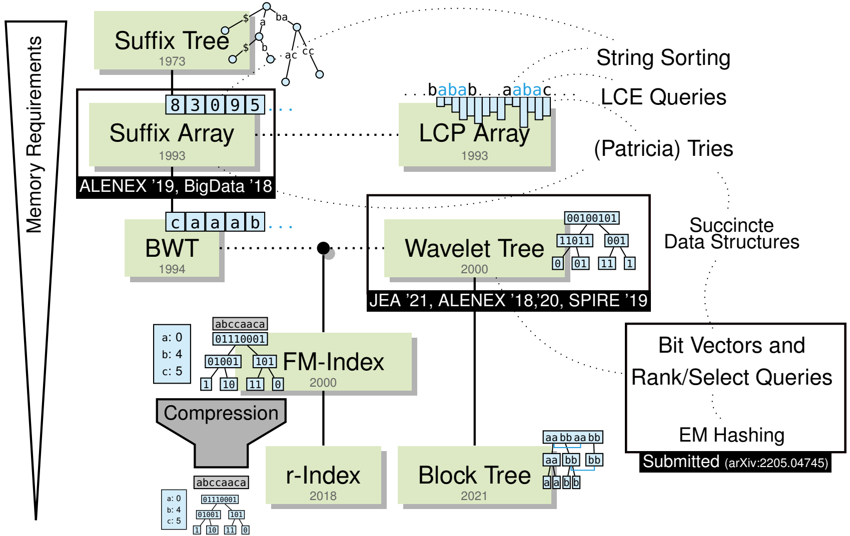
# Recap: Suffix Array

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | a | b | a | b | c | a | b | c | a | b | b | a | $ |
| $SA$ | 13 | 12 | 1 | 9 | 6 | 3 | 11 | 2 | 10 | 7 | 4 | 8 | 5 |
| | $ | a$ | ababcabcabba$ | abba$ | abcabba$ | abcabcabba$ | ba$ | babcabcabba$ | bba$ | bcabba$ | bcabcabba$ | cabba$ | cabcabba$ |

- space: $O(n \log n)$ bits
- space text: $n\lceil \log \sigma \rceil$ bits

- better: index requiring same space as text
- even better: index requiring same space as compressed text

# (Compressed) Text Indices #Ad



# Ψ Function

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**Definition: Ψ Function**

Given a suffix array $SA$ of length $n$,

$$\Psi(i) = SA^{-1}[SA[i] + 1]$$

- $SA[\Psi(i)] = SA[i] + 1$
- where in $SA$ is the suffix $T[SA[i+1]..n)$
- “successor” function

- can be used to obtain suffix array
- can be compressed ℹ currently $O(n \log n)$ bits

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# Replacing *SA* with $\Psi$

- which number does in this example not occur?
  Answer: 3
- how to obtain $SA[i]$ using $\Psi$ **PINGO**

- follow positions until last suffix is found
- last suffix is at position 1
- $n - \#steps$ is $SA$ value
- requires $O(n)$ time

- pattern matching: $O(mn \log n)$ time
- pattern matching with *LCP* and *RMQ*: $O(mn + \log n)$ time

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# Speeding Up Lookups in Ψ (1/2)

- space *SA*: $O(n \log n)$ bits
- space text: $O(n \log \sigma)$ bits
- space compressed suffix array should not more than text

- sample every $\log n$-th *SA* entry
- $O(n / \log n)$ samples of size $O(\log n)$ bits
- total space: $O(n)$ bits

- every $\log n$-th entry in Ψ
- every $\log n$-th step in Ψ
- what is better? **PINGO**

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# Speeding Up Lookups in Ψ (2/2)

- every $\log n$-th entry in Ψ
- every $\log n$-th step in Ψ
- what is better? **PINGO**

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- every $\log n$-th step in Ψ is better
- sampled positions may not be reached in better asymptotic time

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- how much time does recovering *SA* position from Ψ require with sampling? **PINGO**
- answer: $O(\log n)$

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# Structure of $\Psi$

- does $\Psi$ have some structure? **PINGO**

**Lemma: Structure of $\Psi$**

$T[SA[i]] = T[SA[i+1]] \Rightarrow \Psi(i) < \Psi(i+1)$

**Proof (Sketch)**

- $T[SA[i]] \leq T[SA[i+1]]$
- if $T[SA[i]] = T[SA[i+1]]$ then $T[SA[i]+1..n) \leq T[SA[i+1]+1..n)$
- $T[SA[i]+1] = T[\Psi(i)]$
- if suffixes share same character, lexicographical order of suffixes without first character holds

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
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- note that not all increasing intervals belong to the same character
- example on the board

# Compressing Ordered Sequences

## Δ-Encoding

- store difference between entries
- scanning whole sequence up to value when decoding

## Elias-Fano (Lecture 05)

- upper and lower halves
- upper half represented in bit vector $p_i + i$
- lower half plain bit compressed

- using Elias-Fano is bad for large alphabets
- example on the board

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 7 | 10 | 20 | 21 | 22 | 32 |

- 0: `000000`
- 1: `000001`
- 2: `000010`
- 4: `000100`
- 7: `000111`
- 10: `001010`
- 20: `010100`
- 21: `010101`
- 22: `010110`
- 30: `100000`

**upper:** `1110110100011100100`
**lower:** `00 01 10 00 11 10 00 01 10 00`

# Recap: Elias-Fano Coding Space

**Lemma: Elias-Fano Coding**

Given an array containing $n$ distinct integers from a universe $\mathcal{U} = [0, n)$, the array can be represented using

$$n(2 + \log\lceil \frac{u}{n} \rceil) \text{ bits}$$

while allowing $O(1)$ access time and $O(\log \frac{u}{n})$ predecessor/successor time

# Compressing Sparse Ordered Sequences

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- Elias-Fano coding for each increasing interval
- $\sigma$ many
- only every $1/\sigma$-th entry is set (sparse)

- if there are $n$ entries of universe with size $u$
- make entries sparse using $q = u/n$
- for each value $x$ store pair $(x/q, x\%q)$

- $u = 512, n = 8, q = 64$
- $(0, 3, 17, 89, 128, 132, 500, 511)$
- $\{0,0\}, \{0,3\}, \{0,7\}, \{1,25\}$, $\{2,0\}, \{2,4\}, \{7,52\}, \{7,63\}$

- store quotient $(x/q)$ using Elias-Fano
- store remainder $(x\%q)$ plain using $\lceil \log q \rceil$ bits

**Lemma: $\Psi$ with Elias-Fano**

Using Elias-Fano with quotienting, $\Psi$ can be stored using $O(n\sigma)$ bits

- more precise: two additional bits per character

# Simple Compressed Suffix Array

- compute $\Psi$ and store samples of $SA$
- compress $\Psi$ Elias-Fano with quotienting
- binary search on $SA$ by decoding $\Psi$

- space: $O(n \log \sigma)$ space
- query time: $O(m \log^2 n)$

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# Improving Compressed Suffix Arrays [GV05] (1/5)

- improve *SA* lookup to $O(\log \log n)$ time
- divide-and-conquer approach
- storing $\Psi$ only for half of the entries
- recurs for the other half

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| $\Psi$ | - | 1 | 8 | 9 | 10 | 11 | 2 | 6 | 7 | 12 | 13 | 4 | 5 |
| NEW | 13 | 1 | 9 | 3 | 11 | 7 | 5 | 1 | 10 | 6 | 7 | 13 | 4 |

- for which values do we store $\Psi$? **PINGO**

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# Improving Compressed Suffix Arrays (2/5)

- store bit vector marking odd *SA* values
- store only odd *SA* values
- store $\Psi$ for even *SA* values

- store $\Psi$ as before
- Elias-Fano with quotienting
- without sampling

- right half (*SA*) still big
- how to recurs?

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| NEW | 13 | 1 | 9 | 3 | 11 | 7 | 5 | 1 | 10 | 6 | 7 | 13 | 4 |
| *BV* | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

# Improving Compressed Suffix Arrays (3/5)

- *SA* half consists only of odd values
- for value $x$ store $(x-1)/2$
- reversible since all values are odd

- $13, 1, 9, 3, 11, 7, 5$
- $6, 0, 4, 1, 5, 3, 2$

- what do we have here? **PINGO**
- permutation basically a suffix array without text

- recurs on the permutation without explicitly storing it

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| NEW | 13 | 1 | 9 | 3 | 11 | 7 | 5 | 1 | 10 | 6 | 7 | 13 | 4 |
| *BV* | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

# Improving Compressed Suffix Arrays (4/5)

- recurs $\log\log n$ times
- guarantees $O(\log\log n)$ time to obtain $SA$ value
- allows to store final $SA$ within space bounds

## Lemma: Space Final $SA$

Using the divide-and-conquer approach, the final $SA$ requires $O(n)$ bits of space

## Proof (Sketch)

- after $\log\log n$ recursions $SA$ has size $n/2^{\log\log n}$
- each entry requires $\log n$ bits
- total space: $O(n)$ bits

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
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| $BV$ | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

# Improving Compressed Suffix Arrays (5/5)

## Lemma: Decoding Time Improved CSA

An *SA* value can be decoded in $O(\log \log n)$ time using the improved CSA

## Proof (Sketch)

- on each level, odd *SA* values can be decoded using the recursive *SA*
- there are at most $\log \log n$ levels
- on each level, even *SA* values can be decoded in one step, as the next *SA* value is odd

- requires rank and select data structures

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| NEW | 13 | 1 | 9 | 3 | 11 | 7 | 5 | 1 | 10 | 6 | 7 | 13 | 4 |
| *BV* | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

# Conclusion and Outlook

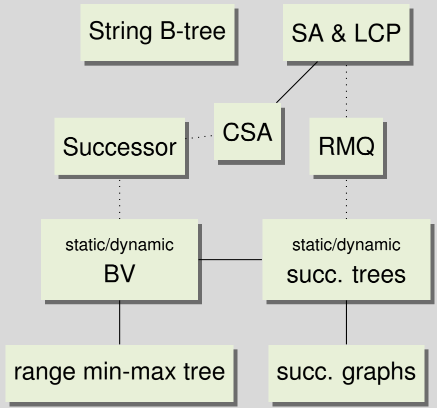
## This Lecture

- compressed suffix array
- note that CSA can be compressed further
- Elias-Fano for sparse sequences

## Next Lecture

- temporal data structures

## Advanced Data Structures



# Bibliography I


[GV05] Roberto Grossi and Jeffrey Scott Vitter. “Compressed Suffix Arrays and Suffix Trees with Applications to Text Indexing and String Matching”. In: *SIAM J. Comput.* 35.2 (2005), pages 378–407. DOI: 10.1137/S0097539702402354.