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Text Indexing

Lecture 08: LZ and BWT Compressed Indeces

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PINGO

https://pingo.scc.kit.edu/669011

Recap: FM-Index and r-Index

- based on backwards-search
- used to answer *rank*-queries on *BWT*

Function *BackwardsSearch*($P[1..n]$, C, *rank*):

```
s = 1, e = n
for i = m, ..., 1 do
    s = C[P[i]] + rank_{P[i]}(s − 1) + 1
    e = C[P[i]] + rank_{P[i]}(e)
    if s > e then
        return ∅
return [s, e]
```

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- build wavelet tree directly on *BWT*
- wavelet tree can be H_0 compressed
- blind to repetitions

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- wavelet tree can be H_0 compressed
- blind to repetitions

r-Index

- many arrays with r entries
- build wavelet tree on one of these arrays
- size in numbers of *BWT* runs r

Function $BackwardsSearch(P[1..n], C, rank)$**:**

$s = 1, e = n$
for $i = m, \ldots, 1$ **do**
  $s = C[P[i]] + rank_{P[i]}(s-1) + 1$
  $e = C[P[i]] + rank_{P[i]}(e)$
  if $s > e$ **then**
    return $\emptyset$
return $[s, e]$

Different Types of Compression

Statistical Coding

- based on frequencies of characters
- results in size $|T| \cdot H_k(T)$
  ℹ *k*-th order empirical entropy
- good if frequencies are skewed
- blind to repetitions
  $|\underbrace{T \ldots T}_{\ell}| \cdot H_k(\underbrace{T \ldots T}_{\ell}) \approx \ell |T| \cdot H_k(T)$

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- references to previous occurrences
- each LZ factor can be encoded in $O(1)$ space
- good for repetitions
- index in this lecture

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- index in this lecture

BWT-Compression

- used in powerful index
- theoretical insight in this lecture

LZ-Compressed Index

Definition: LZ77 Factorization [ZL77]

Given a text T of length n over an alphabet Σ, the **LZ77 factorization** is

- a set of z factors $f_1, f_2, \dots, f_z \in \Sigma^+$, such that
- $T = f_1 f_2 \dots f_z$ and for all $i \in [1, z]$ f_i is
- single character not occurring in $f_1 \dots f_{i-1}$ or
- longest substring occurring ≥ 2 times in $f_1 \dots f_i$

$T =$ abababbbbaba\$

- $f_1 =$ a
- $f_2 =$ b
- $f_3 =$ abab
- $f_4 =$ bbb
- $f_5 =$ aba
- $f_6 =$ \$

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Now

- LZ-compressed replacement for wavelet trees
- *rank* and *access* queries ℹ *select* also supported
- LZ-compression better than H_k-compression

Block Trees [Bel+21] (1/4)

Definition: Block Tree (1/4)

Given a text T of length n over an alphabet of size σ

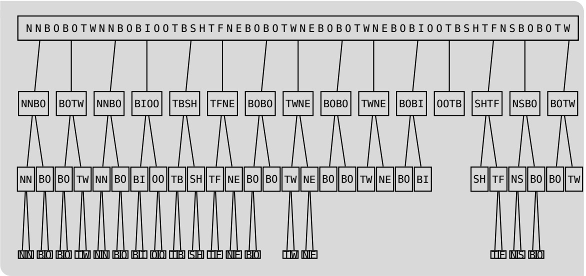
- $\tau, s \in \mathbb{N}$ greater 1
- assume that $n = s \cdot \tau^h$ for some $h \in \mathbb{N}$
  - append \$s until n has this form

A **block tree** is a

- perfectly balanced tree with height h
- that may have leaves at higher levels

such that

- the root has s children,
- each other inner node has τ children



Block Trees (2/4)

Definition: Block Tree (2/4)

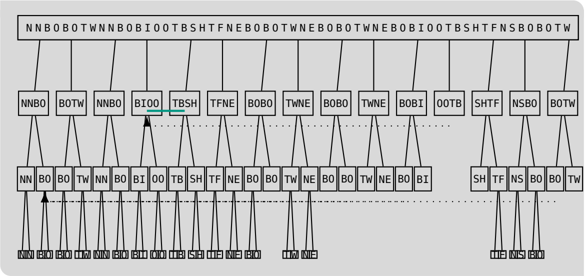
In a block tree, leaves at

- the last level store characters or substrings of T
- at higher levels store special leftward pointer

Each node u

- represents a block B^u
- which is a substring of T identified by a position

The root represents T and its children consecutive blocks of T of size n/s



Block Trees (3/4)

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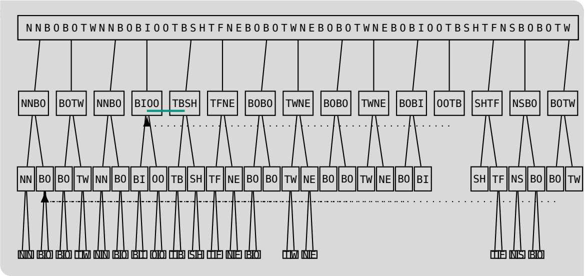
Definition: Block Tree (3/4)

Let ℓ_u be the level (depth) of node u

- the level of the root is 0

Let $B_1, B_2, \ldots$ be the blocks represented at level ℓ_u from left to right

- for any i, B_i and B_{i+1} are consecutive in T
- if $B_i B_{i+1}$ are the leftmost occurrence in T, the nodes representing the blocks are marked



Block Trees (4/4)

Definition: Block Tree (4/4)

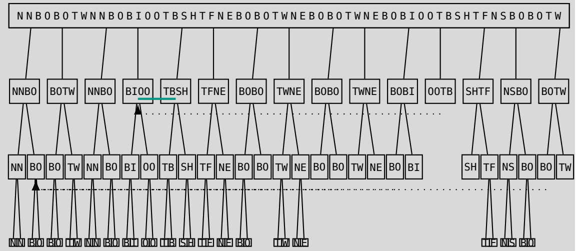
If node u is marked, then

- it is an internal node
- with τ children

otherwise, if node u is not marked, then

- u is a leaf storing
- pointers to nodes v_i, v_{i+1} at the same level
  - that represent blocks B_i and B_{i+1}
  - covering the leftmost occurrence of B^u
- offset to the occurrence of B^u in B_iB_{i+1}

leaves on last level store text explicitly



- $|B^u| = n/(s\tau^{\ell_u - 1})$
- if $|B_u|$ is small enough, store text explicitly
  - $|B^u \in \Theta(\lg_\sigma n)|$

Block Trees are LZ Compressed (1/2)

Lemma: Number of Blocks per Level

The number of blocks in any level > 0 in the block tree is at most $3\tau z$

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Proof (Sketch)

Let $\ell > 0$ be a level in the block tree and

- $C = B_{i-1}B_iB_{i+1}$ a concatenation of three consecutive blocks at level $\ell - 1$
- not containing the end of an LZ factor
- thus a leftwards occurrence in T

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- there are only z LZ factors

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- last level has $O(\tau z)$ blocks with plain text
  - $O(\lg_\sigma n)$ symbols of $\lceil \lg n \rceil$ bits
  - requiring $O(\lg n)$ bits per block

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- $h = \lg_\tau \frac{n \lg \sigma}{s \lg n}$ and $O(s)$ pointers to top level

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- $h = \lg_\tau \frac{n \lg \sigma}{s \lg n}$ and $O(s)$ pointers to top level
- rounding up length adds $\leq Oh\tau$ blocks per level

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Block Trees are LZ Compressed (2/2)

Lemma: Space Requirements of Block Trees

Given a text T of length n over an alphabet of size σ and integers $s, \tau > 1$, a block tree of T has height $h = \lg_\tau \frac{n \lg \sigma}{s \lg n}$. The block tree requires

$$O((s + z\tau \lg_\tau \frac{n \lg \sigma}{s \lg n}) \lg n) \text{ bits of space,}$$

where z is the number of LZ77 factors of T

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- $s = z$ results in a tree of height $O(\lg_\tau \frac{n \lg \sigma}{z \lg n})$
- space requirements $O(z\tau \lg_\tau \frac{n \lg \sigma}{z \lg n})$ bits
- however z not known

Access Queries in Block Trees

- queries are easy to realize
- if not supported directly, additional information can be stored for blocks

Access Query

Given position i return $T[i]$

- follow nodes that represent block containing $T[i]$
- of not marked follow pointer and consider offset
- at leaf, if last level, return character
- else, follow pointer and continue

- time $O(\lg_\tau \frac{n \lg \sigma}{s \lg n})$

- example on the board

Rank Queries in Block Trees

- for each block add histogram $Hist_{B_u}$ for prefix of T up to block (not containing)
- $O(\sigma(s + z\tau \lg_\tau \frac{n \lg n}{s \lg n}) \lg n)$ bits of space

Rank Query

Given position i and character α return $rank_\alpha(T, i)$

- follow nodes that represent block containing $T[i]$
- remember $Hist_{B_u}[\alpha]$
- of not marked follow pointer and consider offset
- at leaf, if last level, compute local rank binary rank for each character
- else, follow pointer and continue

- time $O(\lg_\tau \frac{n \lg \sigma}{s \lg n})$

- example on the board

Construction of Block Trees

$O(n)$ Working Space

- build Aho-Corasick automaton for containing all pairs of consecutive unmarked blocks
- identify unmarked blocks on next level
- $O(n(1 + \lg_{\tau} \frac{z}{s}))$ time and $O(n)$ space

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Pruning

- size of block tree can be reduced further
- some blocks not necessary
- those blocks can easily be identified

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$O(s + z\tau)$ Working Space

- replace Aho-Corasick automaton with Karp-Rabin fingerprints
- validate if matching fingerprints due to matching strings ℹ Monte Carlo algorithm
- $O(n(1 + \lg_\tau \frac{z}{s}))$ expected time and $O(n)$ space
- only expected construction time!

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- queries very fast in practice
- construction very slow in practice
- good topic for thesis ☺

Relation Between BWT Runs and LZ Factors [KK20] (1/3)

Let T be a text, then

- $r(T)$ is number of *BWT* runs of T
- $z(T)$ is number of LZ77 factors of T

Definition: Burrows-Wheeler Transform [BW94]

Given a text T of length n and its suffix array SA, for $i \in [1, n]$ the **Burrows-Wheeler transform** is

$$BWT[i] = \begin{cases} T[SA[i]-1] & SA[i] > 0 \\ \$ & SA[i] = 0 \end{cases}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
LCP	0	0	1	2	2	5	0	2	1	1	4	0	3
BWT	a	b	$	c	c	b	b	a	a	a	a	b	b

Relation Between BWT Runs and LZ Factors (2/3)

Lemma: Number of BWT Runs

Let T be a text of length n, then

$$r(T) \in O(z(T) \lg^2 n)$$

- $LCP[i]$ is irreducible if $i = 1$ or $BWT[i] \neq BWT[i-1]$
- number of irreducible LCP-values is $r(T)$

Lemma: Sum of Irreducible LCP-Values

The number of irreducible LCP-Values in $[\ell, 2\ell]$ is in $O(z\ell \lg n)$

- $r(T)$ is number of irreducible LCP-values
- apply lemma for $[2^i, 2^{i+1})$ for $i \in [0, \lfloor \lg n \rfloor]$
- number of $LCP[i] = 0$ entries is $\sigma \leq z$

Relation Between BWT Runs and LZ Factors (3/3)

Lemma: Number of Occurrences of Substrings

For any $\ell > 1$, the number of distinct substrings of T of length ℓ is $\leq z\ell$

Proof (Sketch)

- consider any substring of length $\ell > 1$
- if substrings is contained in LZ factor, there is previous occurrence
- distinct substrings overlap LZ factors
- there are at most ℓ substring per end of LZ factor

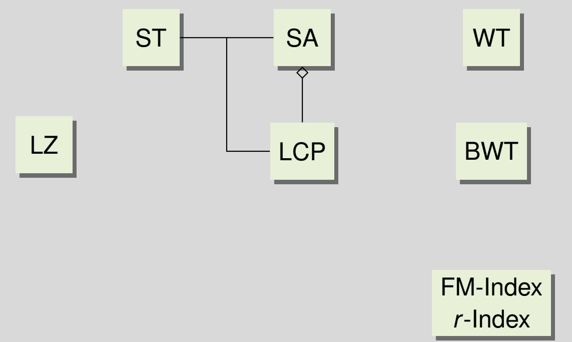
- use number of distinct substrings
- to show that the number of irreducible LCP-values
- is limited as stated in lemma

Conclusion and Outlook

This Lecture

- block trees
- $r \in O(z \lg^2 n)$

Linear Time Construction



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Open Questions

- efficient block tree construction
- linear time block tree construction

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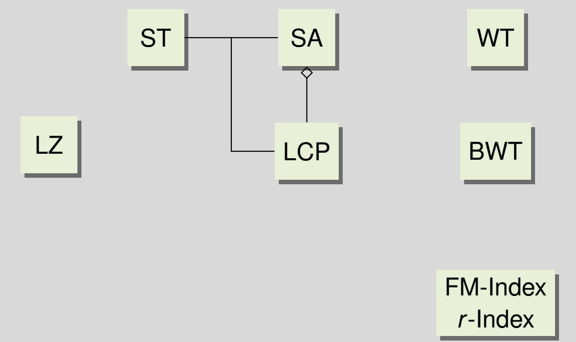

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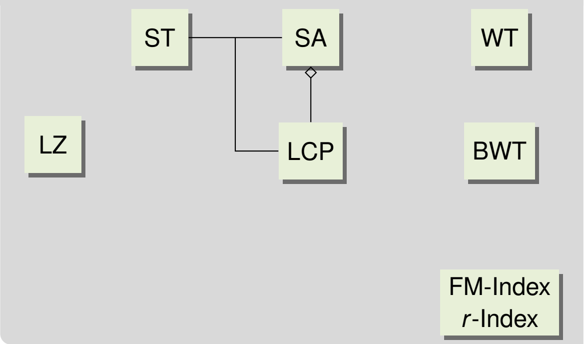
Open Questions

- efficient block tree construction
- linear time block tree construction

Next Lecture

- suffix array construction in different models of computation

Linear Time Construction



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